Reduced MHD — Brief

See Strauss, '76 (PoF) for full details

→ the point: – strong $\langle \underline{B} \rangle$, ~ straight

– low frequency ($\omega < \omega_{MS}$)

– $\langle B \rangle$ ~ unperturbed

– $\nabla \cdot \underline{V} = 0$

$\underline{\nabla} \cdot \underline{V} = 0 \Rightarrow$ 2 component $\underline{V}$

$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow$ 2 components $\underline{B}$

$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} \simeq 0 \Rightarrow \underline{V}_\perp = +\frac{c}{B} \underline{E}_\perp \times \vec{z}$

$\underline{E}_\perp = -\underline{\nabla}_\perp \phi \cancel{- \frac{1}{c} \frac{\partial A_\perp}{\partial t}}$ $E_{B_\parallel} \to 0$

$\underline{V}_\perp = -\frac{c}{B} \underline{\nabla}_\perp \phi \times \vec{z}$

$\underline{\nabla} \times \underline{A} = 0$

$\underline{\ }_\perp = \nabla A_\parallel \times \vec{z}$ $\quad B_\perp = \nabla A_\parallel \times \vec{z}$

Then,

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = \eta \underline{J}$$

$$E_{\parallel} = -\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \nabla_{\parallel}\phi = \eta J_{\parallel}$$

$$\underline{B} = B_0 \hat{z} + \underline{B}_{\perp}$$

$\Rightarrow$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \frac{(B_0\hat{z} + \tilde{B}_{\perp})\cdot\nabla\phi}{|\underline{B}_0\hat{z} + \tilde{\underline{B}}_{\perp}|} = \eta J_{\parallel}$$

$\therefore$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \tilde{B}_{\perp}\cdot\nabla\phi = \eta J_{\parallel}$$

and

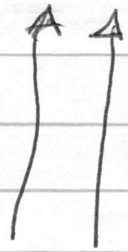
$$\boxed{-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \underline{\nabla}A_{\parallel}\times\hat{z}\cdot\nabla\phi = \eta J_{\parallel}}$$

$\Rightarrow$

$$\boxed{\frac{\partial \psi}{\partial t} + \underline{v}\cdot\nabla\psi = \partial_z\phi + \eta\nabla^2\psi}$$

Reduced Ohm's Law

Now,



strong field
+
$v_\perp$ only → set by ϕ

So, $\dfrac{\underline{B}_0 \cdot \underline{\nabla}\times\underline{V}}{|B_0|} = \hat{z}\cdot\underline{\nabla}\times\underline{V}$ → is the key to dynamics

Now, $\rho\left(\dfrac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla}P + \dfrac{\underline{J}\times\underline{B}}{c}$

$\underline{v}\cdot\underline{\nabla}\,\underline{v} = \underline{\nabla}\dfrac{\underline{v}^2}{2} - \underline{v}\times\underline{\omega}$

$\rho\left(\dfrac{\partial v}{\partial t}\right) = -\underline{\nabla}\left(P + \rho\dfrac{\underline{V}^2}{2}\right) + \rho\,\underline{v}\times\underline{\omega} + \dfrac{\underline{J}\times\underline{B}}{c}$

$\rho \equiv \rho_0$ (incompr/Boussinesq)

$\dfrac{\partial \underline{v}}{\partial t} = -\underline{\nabla}\left(\dfrac{P}{\rho} + \dfrac{\underline{v}^2}{2}\right) + \underline{v}\times\underline{\omega} + \dfrac{\underline{J}\times\underline{B}}{c\rho}$

$= -\underline{\nabla}\left(\dfrac{P}{\rho} + \dfrac{B^2}{8\pi\rho} + \dfrac{v^2}{2}\right) + \underline{v}\times\underline{\omega} + \dfrac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho}$

$= -\underline{\nabla}\left(P/\rho + \dfrac{B^2}{8\pi} + \dfrac{v^2}{2}\right) + \underline{v}\times\underline{\omega} + \dfrac{B_0\,\partial_z \tilde{B}_\perp}{4\pi\rho} + \dfrac{\tilde{B}_\perp\cdot\underline{\nabla}_\perp\tilde{B}_\perp}{4\pi\rho_0}$ (no B_0 curv) so $\underline{\nabla} \times$

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{\omega} + \frac{B_0}{4\pi\rho_0} \partial_z \underline{\nabla} \times \underline{\tilde{B}}_\perp$$

$$+ \frac{\underline{\tilde{B}}_\perp}{4\pi\rho_0} \cdot \nabla_\perp \left(\underline{\nabla} \times \underline{\tilde{B}}_\perp \right)$$

$$= -\underline{V} \cdot \underline{\nabla} \omega + \underline{\omega} \cdot \underline{\nabla} \underline{V} + \frac{B_0}{4\pi\rho} \partial_z \underline{\nabla} \times \underline{\tilde{B}}_\perp$$

$$+ \frac{\underline{\tilde{B}}_\perp}{4\pi\rho} \cdot \nabla_\perp \left(\nabla \times \tilde{B}_\perp \right)$$

$\hat{z} \cdot ( ) \Rightarrow$

$$\Rightarrow \frac{d}{dt} \omega_z = \cancel{\underline{\omega} \cdot \underline{\nabla} V_z} + \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_{z\parallel} + \frac{\nabla A_\parallel \times \hat{z}}{4\pi\rho_0} \cdot \nabla \tilde{J}_\parallel$$

$$\omega_z \rightarrow \nabla^2 \phi$$

so

$$\boxed{\frac{d}{dt} \nabla^2 \phi = \partial_t \nabla^2 \phi + \underline{V} \cdot \underline{\nabla} \nabla^2 \phi = \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_z + \frac{\underline{\tilde{B}}}{4\pi\rho_0} \cdot \nabla \tilde{J}_z}$$

$\rightarrow$ Vorticity eqn!

Alternative Approach:

①

$\hat{n} \cdot (\underline{E} = \eta \underline{J})$

→ as before!

②

$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$, continuity!

$\rho = (n_i - n_e) e$

and QN ⇒

$\boxed{\underline{\nabla} \cdot \underline{J} = 0}$ → generic!

⇒

$\underline{\nabla}_\perp \cdot \underline{J}_\perp = -\nabla_\parallel J_\parallel$

$$\underline{J}_\perp = (\underbrace{n_i \underline{V}_E - n_e \underline{V}_E) e}_{\underline{E}\times\underline{B} \text{ current, cancels}} + \underbrace{n_i e \, \underline{V}_{pol}}_{\text{polarization current} \to \text{ions } (m_i \gg m_e)}$$

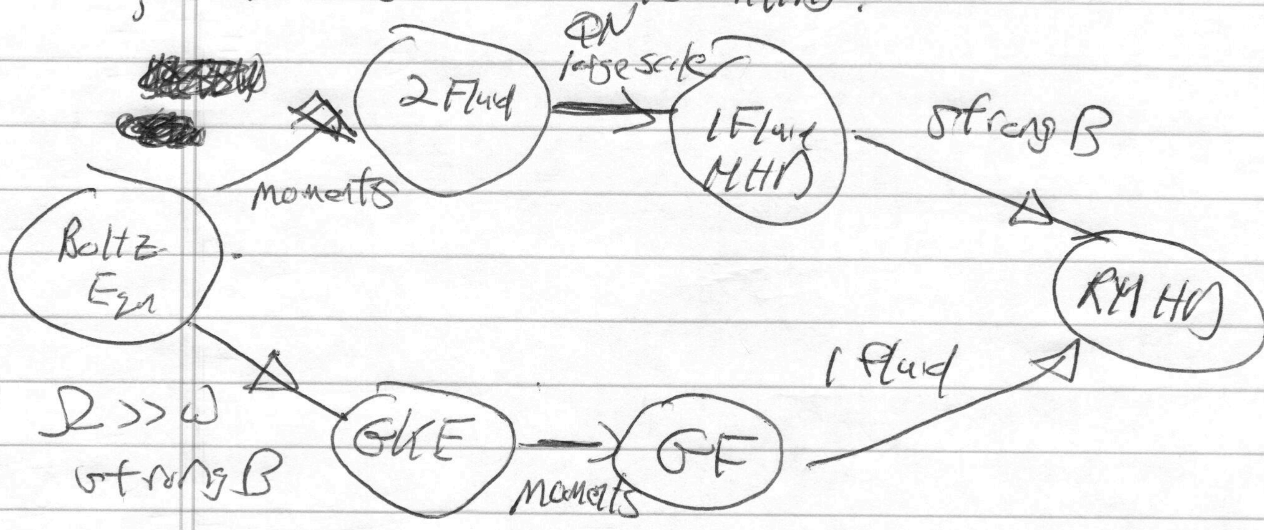
$$\underline{\nabla}_\perp \cdot (n_0 \underline{V}_{pol}) = \frac{-1}{e} \nabla_\parallel J_\parallel$$

$$= -1/e \left(\partial_z \tilde{J}_\parallel + \tilde{\underline{B}}_\perp \cdot \underline{\nabla}_\perp \tilde{J}_\parallel \right)$$

and back to vorticity eqn.!

⇒ can extend to H-W, H-M, 3 field, ITG....

→ Now, can relate routes to RMHD:



so can come to RMHD by different orders of strong field and fluid approx.